The Relation of the Earth's Free Precessional Nutation to its Resistance against Tidal Deformation.

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The modern investigation of the wandering of the Earth's axis of rotation, considered as a physical problem relating to the actual non-rigid Earth, may be said to have been initiated in Lord Kelvin's address to the Physical Section of the British Association in 1876. After referring* to the scrutiny of the recorded observations of change of latitudes, conducted by Peters in 1841 and independently by Maxwell in 1851, in search of the regular Eulerian free period of 306 days which would belong to a rigid Earth, with negative results, he insisted that the irregular motions brought out in these analyses are not merely due to instrumental imperfections, but represent true motions of the Pole, due to displacement of terrestrial material. For example, he estimates that existing shifts of material, of meteorological type, are competent to produce displacements of the axis of rotation ranging from 1/2 to 1/20 of a second of arc. A sudden shift of material on the Earth will not at once affect the axis of rotation, but will start it into motion round the altered axis of inertia, with a period of 306 days if the Earth were rigid, which will go on displacing the Pole until it is damped out by the frictional effects of the tidal motions thus originated. A radius of rotation of 1 second of arc would raise an ocean tide of the same period as the rotation, having as much as 11 cm. of maximum rise and fall. Thus the motion of the Pole is to be considered as continually renewed by meteorological and other displacements, as it is damped off by tidal and elastic friction; it was therefore, perhaps, not to be expected that it would show much periodicity, though the movements were eminently worthy of close investigation. Their nature was examined more closely by Newcomb at Kelvin's request; but not much more had been done regarding their cause when Chandler announced that the records of changes of latitude did actually indicate a period of precession—of 427 days, however, instead of the Eulerian period of 306 days, which, if any, had previously been taken for granted. Soon after, in 1890, observations were organised systematically by the International Geodetic Union on the motion of Prof. Foerster, of Berlin; and already, in 1891, he was able to inform Lord Kelvin that a comparison

* Reprint in 'Popular Lectures and Addresses,' vol. 2, see pp. 262—272.

of European observations with synchronous ones made at Honolulu gave direct proof of his conclusion of 1876 (*supra*), "that irregular movements of the Earth's axis to the extent of half a second may be produced by the temporary changes of sea level due to meteorological causes."*

In the following year the synchronous observations had already indicated periodicity, apparently in about 385 days, considerably less than Chandler's estimate, which, however, longer observation has since confirmed substantially. Lord Kelvin remarks in his next annual address as follows†:—"Newcomb, in a letter which I received from him last December, gave what seems to me undoubtedly the true explanation of this apparent discrepance from dynamical theory, attributing it to elastic yielding of the Earth as a whole. He added a suggestion, especially interesting to myself, that investigation of the periodic variations of latitude may prove to be the best means of determining approximately the rigidity of the Earth. As it is, we have now for the first time what seems to be a quite decisive demonstration of elastic yielding of the Earth as a whole, under the influence of a deforming force, whether of centrifugal force round a varying axis, as in the present case, or of tide-generating influences of the Sun and Moon, with reference to which I first raised the question of elastic yielding of the Earth's material many years ago." But "when we consider how much water falls on Europe and Asia during a month or two of rainy season, and how many weeks or months must pass before it gets to the sea, and where it has been in the interval, and what has become of the air from which it fell, we need not wonder" that the amplitudes of the polar wanderings "should often vary by 5 or 10 metres in the course of a few weeks or months."

It will be recalled that the main object of the original calculations of Lord Kelvin, which assigns to the Earth as a whole an effective rigidity of the same order as that of steel, was to combat the view then prevalent which assumed for the Earth a fluid interior. Even a solid shell of very considerable thickness, enclosing a fluid core, was thus ruled out, unless its materials were preternaturally rigid; and it is clear that placing a solid core in the middle of the fluid interior cannot affect this conclusion so long as an equilibrium theory is applicable, *i.e.*, so long as the layer of fluid material is not so thin or viscous as to prevent its adjusting itself immediately by flow to the alternating tidal stresses impressed upon it from its solid walls. By passing to the other limit, and thus taking it so thin that the outer shell

* 'Presidential Address R.S.,' Nov. 30, 1891; 'Popular Lectures ...,' vol. 2, p. 504. Lord Kelvin's investigations up to 1876 are collected in 'Math. and Phys. Papers,' vol. 3, especially pp. 312—350.

† 'Presidential Address R.S.,' Nov. 30, 1892; *loc. cit.*, p. 525.

practically rides on the solid nucleus, but without effective tangential stress-connection, we obtain a hypothesis to which this objection does not apply.

In a brief note in 'Monthly Notices R.A.S.' this year (1892), Newcomb showed, by a general estimate, that the effect of elastic yielding is competent to prolong the free period to about the amount required by observation. A formal mathematical discussion on the bases of calculation of the elastic deformation of a homogeneous sphere was first given by Mr. S. S. Hough, now H.M. Astronomer at the Cape of Good Hope, in a memoir on "The Rotation of an Elastic Spheroid," in 'Phil. Trans.,' 1896.

He concluded that the Chandler free period required an effective rigidity of the whole Earth of the order of that of steel, agreeing with Lord Kelvin's previous estimates from tidal phenomena; and his result seems to have been substantially confirmed by more recent calculations, giving for the average effective rigidity estimates derived from various possible hypotheses and simplifying assumptions ranging between extreme values 17×10^{11} and $4{\cdot}4 \times 10^{11}$, while Hough's estimate was put at $8{\cdot}98 \times 10^{11}$. This shows an even striking degree of agreement in calculations necessarily vague on account of the unknown constitution of the Earth's interior, especially in so far as observations of the equilibrium tides of long periods, and of the deviation of sea level due to tidal attraction which is essentially the same thing, lead to results of the same order as those of free precessional rotation.* It, indeed, suggests, as we shall actually recognise, that this internal terrestrial constitution really is not involved in these various phenomena, except in the common feature of determining the surface effects arising from a given tidal or rotational stress.† The key to the matter, from the general point of view, is contained in the remark of Hough that the free precession of the yielding Earth is the same as that of a rigid one of the shape that would result when the bulging arising from the centrifugal force of diurnal rotation is removed. It is not difficult to show, from geometrical considerations regarding momentum,‡ that this result is general, and extends to an Earth of any degree of heterogeneity or plasticity. The argument may be reproduced in analytical form and rather wider scope, from another place§ (with definition of I rewritten), as follows:—

Let ω be the angular velocity of the Earth about the instantaneous axis,

* *Cf.* Prof. A. E. H. Love, 'Roy. Soc. Proc.,' *supra*, p. 73. To this paper I am indebted for information as to results of recent calculations.

† The identity of these two types runs through the discussions in Thomson and Tait's 'Natural Philosophy.'

‡ 'Proc. Camb. Phil. Soc.,' May, 1896, p. 185.

§ E. H. Hills and J. Larmor, "The Irregular Movement of the Earth's Axis of Rotation," 'Monthly Notices R.A.S.,' Nov., 1906, p. 24.

ω_1, ω_2, ω_3 its components referred to the principal axes in the configuration that the Earth would have if the motion were steady. The Earth is deformed from this configuration by the inequality of centrifugal force due to the deviation of the instantaneous axis from the principal axis, with which it would coincide if the motion were steady. This deforming force is the resultant of the centrifugal force, directed outwards from the instantaneous axis, and the reversed centrifugal force, directed inwards towards the principal axis in question. A linear law of elasticity applies to the small resultant of these two forces. If the same law applied to the two forces separately, the reversed centrifugal force would change the moments of inertia A, B, C to certain values A′, B′, C′, which might, under simplifying hypotheses, be calculated from the theory of the deformation of an elastic sphere; and the centrifugal force directed outwards from the instantaneous axis would produce a certain change of density at each internal point, and would raise a certain protuberance on the surface, which might be calculated by the same theory. Let I denote the moment of inertia (about the instantaneous axis) of a mass arranged as specified by this change of density and this protuberance. The instantaneous axis is a principal axis of this mass, and therefore the contributions of this mass to the components of moment of momentum are $I\omega_1$, $I\omega_2$, $I\omega_3$. The complete expressions for the components h_1, h_2, h_3 of moment of momentum are therefore

$$h_1 = A'\omega_1 + I\omega_1, \quad h_2 = B'\omega_2 + I\omega_2, \quad h_3 = C'\omega_3 + I\omega_3.$$

The equations of motion referred to the rotating axes are of the well-known vector type,

$$dh_1/dt - h_2\omega_3 + h_3\omega_2 = L.$$

When A and B are equal, the third of them is

$$\frac{d}{dt}(C\omega_3) = N,$$

where C is the effective moment of inertia $C'+I$: when N is null ω_3 is thus constant, say Ω, up to the first order. The other two equations are

$$\frac{d}{dt} \cdot (A'+I)\,\omega_1 + (C'-B')\,\Omega\omega = L,$$

$$\frac{d}{dt} \cdot (B'+I)\,\omega_2 - (C'-A')\,\Omega\omega_1 = M,$$

which in the case of approximate symmetry involve a free period $2\pi(A'+I)/(C'-A')\,\Omega$, and similarly in the general case, thus depending only on A′, B′, C′ when I is small.

The result is that the period of the free precession is not $C/(C-A)$ days, as it would be for a rigid Earth, but approximately $C/(C'-A')$, where the

denominator is that difference of principal moments of inertia which would remain after the imposition of a bodily forcive having as potential

$$W = -\tfrac{1}{2}\omega^2 r^2 \sin^2\theta = -\tfrac{1}{3}\omega^2 r^2 (1 - P_2),$$

namely, that of the centrifugal force reversed, P_2 representing the zonal harmonic $\frac{1}{2}(3\cos^2\theta - 1)$.

The first part of W, the term $-\frac{1}{3}\omega^2 r^2$, corresponds to slight contraction of volume, which is immaterial as regards the desired quantity $C' - A'$. The other part, $\frac{1}{3}\omega^2 r^2 P_2$, will produce an extension, of the same harmonic type as itself, along the polar axis, which will in turn alter the potential of the Earth's attraction at its own surface by $k \cdot \frac{1}{3}\omega^2 r^2 P_2$, where the value of k depends on its effective resistance to deformation. Moreover the Earth's potential is at distant points, by Laplace's formula,

$$V = \gamma\left(\frac{E}{r} + \frac{A+B+C-3I}{2r^3} + \ldots\right),$$

which gives

$$V = \gamma\left(\frac{E}{r} - \frac{C-A}{r^3} P_2 + \ldots\right)$$

in the present special case; and if, as in the actual circumstances, further harmonics do not occur to sensible amount, this expression holds right up to the Earth's surface. The free surface, of ellipticity ϵ, is

$$\begin{aligned} r &= a(1 + \epsilon \sin^2\theta) \\ &= \mathrm{a}(1 - \tfrac{2}{3}\epsilon P_2), \end{aligned}$$

where $\mathrm{a} = a(1 + \frac{2}{3}\epsilon)$. The value of ϵ is determined by the constancy over the ocean surface of the total potential $V - W$, as $-W$ is the potential of the centrifugal force, viz., of

$$\gamma\left\{\frac{E}{\mathrm{a}}(1 + \tfrac{2}{3}\epsilon P_2) - \frac{C-A}{a^3} P_2\right\} + \tfrac{1}{3}\omega^2 a^2 (1 - P_2);$$

whence, equating to zero the coefficient of P_2,

$$\tfrac{2}{3} g a \left(\epsilon - \frac{\omega^2 a}{2g}\right) - \frac{\gamma}{a^3}(C - A) = 0,$$

thus deriving from data of the distribution of gravity, or of the form of the Earth's surface, the value of $C - A$, which determines the astronomical precession. Again, if taking off the centrifugal force would change $C - A$ to $C' - A'$, it would alter V by $\gamma r^{-3}\{(C-A) - (C'-A')\} P_2$, which must, according to the above specification of k, be equal to $k \cdot \frac{1}{3}\omega^2 r^2 P_2$. Thus

$$\begin{aligned} \frac{C'-A'}{C-A} &= 1 - \frac{\frac{1}{3}k\omega^2 a^5 \gamma^{-1}}{C-A} \\ &= 1 - \frac{k\,\omega^2 a/2g}{\epsilon - \omega^2 a/2g}. \end{aligned}$$

Hence, if τ is the periodic time of actual free precession and τ_0 is what it would be if the Earth were rigid,

$$1-\frac{\tau_0}{\tau}=k\frac{\omega^2 a}{2g}\Big/\left(\epsilon-\frac{\omega^2 a}{2g}\right).$$

This is the formula (15) in Prof. Love's paper before referred to; it is there deduced from a hypothesis of concentric spheroidal stratification of the Earth's interior, after the manner of Laplace. We have found that, like Clairaut's formula for gravity, this relation is independent of any hypothesis as to the Earth's internal structure, except such as is involved in the definition and value of k.

As $\omega^2 a/g$ is 1/289 and τ is found to be 428 days, and τ_0 is 306 days, this relation makes k equal to 4/15.

The values of k corresponding to various moduli of rigidity and compressibility of the Earth considered as a homogeneous globe might perhaps be deduced and tabulated for comparison, from Lord Kelvin's and similar elastic analysis.

The height of the long-period equilibrium tides provides different data; corresponding to an extraneous tide-producing potential W_2 of this type, the absolute rise of the water is $(1+k)\,W_2/g$, from which has to be subtracted hW_2/g for the rise of the solid Earth due to this tide-producing potential, thus leaving a factor $1+k-h$ for the relative tide which alone can be the subject of observations. The reductions of tidal data for the Indian Ocean gave Kelvin and G. H. Darwin the value 2/3 for this factor, which is confirmed by more recent discussions: the observations of Hecker with a horizontal pendulum at the bottom of a well, which obviously determine the same thing, viz., the change of level due to tide-producing potential, concur in a remarkable manner. Thus $h = 3/5$.

These values of k and h, as defined in the last paragraph, would not be independent for a homogeneous incompressible globe: they would, in general, require for their consistency both elasticity of volume and of form. The phenomena of free precession give the value of k, but with reference to compression along the polar axis; those of tidal change of level give the value of $h-k$, or rather its mean value, with reference to compression along axes in the neighbourhood of the equator.* This statement is the purest and simplest expression of the information relating to the solid Earth's resistance to deforming forces that the data of periodic change of latitude

* *Cf.* Prof. Love, *supra*, p. 81, to whom this proposition is substantially due, having been reached by him through analysis appropriate to a centrically stratified body. The quantities h and k, in other notation, enter essentially into the tidal discussions by Kelvin and Darwin in Thomson and Tait's 'Nat. Phil.'

and of equilibrium (*i.e.*, long-period) tides can supply, prior to any hypothesis regarding the internal distribution and the effective elasticity or plasticity of its materials.

[*Added February* 2.—It has been remarked above, after Lord Kelvin, that a sudden shift of material from one part of the Earth's surface to another would alter the position of the principal axis of inertia round which the free precession of the Earth's axis of rotation takes place, and thus cause a sharp bend in the path of the Pole. If the shift were merely local, such as an earthquake may be expected to produce, the effect would be inappreciable. The connection of sharp curvature in the path of the Pole with seismic disturbance, if it really exists, would thus be indirect, the earthquake being itself started possibly by the slight changes, meteorological or other, of distribution of surface load, which are indicated by the disturbance of the free precession.

But it is to be noticed that a *submarine* seismic subsidence, if uncompensated by adjacent elevation, or *vice versâ*, would be competent to produce sensible direct disturbance of the path of the Pole; for water would have to flow, in part from distant regions, to fill up the defect of level thus produced. The same would be true for earthquake subsidence near coast lines, if it is compensated by rise of the land. In reply to an inquiry on this subject, Prof. Milne writes as follows:—"When a very large earthquake occurs on land, we find vertical and lateral displacements of, let us say, 20 feet, along lines which may be one or two hundred miles in length. The majority of big earthquakes, however, are sub-oceanic in their origin, along lines parallel to mountain ridges, as, for example, at the bottom of the trough which runs parallel to the Andes. The mass movement appears to result in the deepening of the trough and the rise of the coast line. We have measurements where depth has increased as much as 200 fathoms: see 'Brit. Assoc. Seismic Report,' 1897, for a number of these measurements."

An estimate of the effect of such displacements is easily made. Thus, an uncompensated subsidence of the ocean floor, of volume corresponding to a fall of one foot over a thousand miles square, in middle latitudes, would produce* a direct shift in the Pole of rotation amounting to about one-eighth of a second of arc; and at the same time the Pole of the principal axis of inertia, round which the 428-day precession of the axis of rotation takes place, would be displaced in the opposite direction through an angle of the same order of magnitude.

In connection with the possibility of irregularity in the Earth's diurnal

* *Loc. cit.*, 'Monthly Notices R.A.S.,' Nov., 1906, p. 26.

rotation due to causes of this kind, similar considerations arise.* A slight subsidence, due to shrinkage around the equator, unless it extended downward a long way toward the Earth's centre, would have negligible direct effect on the moment of inertia and, therefore, on the length of the day; but if it were under sea it would involve transference of water from regions nearer the Earth's axis, in order to make up the deficiency, and if the equatorial regions were all under water, a contraction of 50 cm. in equatorial radius would in this way alter the length of the year by an amount of the order of half a second of time, which would be astronomically of high importance.]

Notes on Observations of Sun and Stars in some British Stone Circles. Fourth Note.—The Botallek Circles, St. Just, Cornwall.

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Borlase, in his "Antiquities of Cornwall" (p. 199), published in 1769, refers to what he terms "the curious cluster" of circles at Botallek, the seeming confusion of which led him to write "I cannot but think that there was some mystical meaning, or, at least, distinct allotments to particular uses."

Fortunately for science, he accompanies his account with a plan evidently carefully prepared (fig. 1), which is now the only thing that remains; every stone has been utilised in building an engine house, or in other ways. Only the site is shown on the ordnance map.

As the "cluster" of circles exceeds in elaboration anything of the kind with which I am acquainted, it was of great interest to see if anything could be made of it in the light of other researches in Cornwall already referred to in previous communications to the Royal Society.† The first point of

* Lord Kelvin, *loc. cit.*, § 38.

† 'Roy. Soc. Proc.,' A, vol. 76, 1905, p. 177; A, vol. 77, 1906, p. 465; A, vol. 80, 1908, p. 285.